A new two-component integrable system with peakon and weak kink solutions

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Abstract

A new two-component system with cubic nonlinearity:

$$\begin{cases} m_t = bu_x + \frac{1}{2}[m(uv - u_xv_x)]_x - \frac{1}{2}m(uv_x - u_xv), \\ n_t = bv_x + \frac{1}{2}[n(uv - u_xv_x)]_x + \frac{1}{2}n(uv_x - u_xv), \\ m = u - u_{xx}, \\ n = v - v_{xx}, \end{cases}$$

where b is an arbitrary constant, is proposed in this paper. This system is shown integrable with its Lax pair, bi-Hamiltonian structure, and infinitely many conservation laws. As a special case of $b = 0$, $v = 2$, a new Lax pair is given for the standard Camassa-Holm equation: $m_t + m_xu + 2mu_x = 0, m = u - u_{xx}$. In the case of $b = 0$, the peaked soliton (peakon) and multi-peakon solutions are studied. In particular, the two-peakon dynamical system is explicitly solved and their collisions are investigated in details. In the case of $b \neq 0$, the weak kink solution is also discussed.

Keywords: Integrable system, Lax pair, Peakon, Weak kink.
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1 Introduction

In recent years, the Camassa-Holm (CH) equation [1]

$$m_t - bu_x + 2mu_x + m_x u = 0, \quad m = u - u_{xx}, \tag{1}$$

where b is an arbitrary constant, derived by Camassa and Holm [1] as a shallow water wave model, has attracted much attention in the theory of soliton and integrable system. The CH equation was implicitly implied in the work of Fuchssteiner and Fokas on hereditary symmetries as a very special case [2]. Since the work of Camassa and Holm [1], more diverse studies on this equation have been remarkably developed [3]-[13]. The most interesting feature of the CH equation (1) is to admit peaked soliton (peakon) solutions in the case of $b = 0$. A peakon is a weak solution in some Sobolev space with corner at its crest. The stability and interaction of peakons were discussed in several references [14]-[18].

In addition to the CH equation, other integrable models with peakon solutions have been found [19]-[29]. Among these models, there are two integrable peakon equations with cubic nonlinearity, which are

$$m_t = bu_x + \left[m(u^2 - u_x^2)\right]_x, \quad m = u - u_{xx}, \tag{2}$$

and

$$m_t = u^2 m_x + 3uu_x m, \quad m = u - u_{xx}. \tag{3}$$

Equation (2) was proposed independently by Fokas (1995) [24], Fuchssteiner (1996) [25], Olver and Rosenau (1996) [4], and Qiao (2006) [26] where it was recognized as a nonlinear water wave model. Because of the work by Qiao [26] about its generation from the two-dimensional Euler equation, Lax pair, M/W-shaped soliton, and peaked/cusped solitons, equation (2) regains remarkable attention in the study of peakon and integrable cubic nonlinear system ten years after their work [24, 25, 4]. Equation (2) is the first cubic nonlinear integrable system possessing peakon solutions. Recently, the peakon stability of equation (2) in the case of $b = 0$ was worked out by Gui, Liu, Olver and Qu [30]. In 2009, Novikov [29] derived another cubic equation, which is equation (3), from the symmetry approach, and Hone and Wang [28] gave its Lax pair, bi-Hamiltonian structure, and peakon solutions. Very recently [31], we derived the Lax pair, bi-Hamiltonian structure, peakons, weak kinks, kink-peakon interactional and smooth soliton solutions for the following integrable equation with both quadratic and cubic nonlinearity:

$$m_t = bu_x + \frac{1}{2}k_1\left[m(u^2 - u_x^2)\right]_x + \frac{1}{2}k_2(2mu_x + m_x u), \quad m = u - u_{xx}, \tag{4}$$

where b, k_1, and k_2 are three arbitrary constants.

It is very interesting for us to study the multi-component integrable generalizations of peakon equations. In [4, 32, 33], the authors proposed the two-component generalizations of the CH equation. A two-component integrable extension of the cubic nonlinear equation (2) with $b = 0$ was introduced in [34]. In this paper, we propose the following two-component system with cubic nonlinearity

$$\begin{cases} m_t = bu_x + \frac{1}{2}[m(uv - u_x v_x)]_x - \frac{1}{2}m(uv_x - u_x v), \\ n_t = bv_x + \frac{1}{2}[n(uv - u_x v_x)]_x + \frac{1}{2}n(uv_x - u_x v), \\ m = u - u_{xx}, \\ n = v - v_{xx}, \end{cases} \tag{5}$$

where b is an arbitrary constant. This system is reduced to the CH equation (1), the cubic CH equation (2), and the generalized CH equation (4) as $v = 2$, $v = 2u$, and $v = k_1 u + k_2$, respectively. We prove integrability of this system by providing its Lax pair, bi-Hamiltonian structure, and infinitely many conservation laws. As a special case of $b = 0$, $v = 2$, a new Lax pair is provided for the CH equation. In the case of $b = 0$, we show that this system admits the single-peakon of traveling wave solution as well as multi-peakon solutions. In particular, the two-peakon dynamic system is explicitly solved and their collisions are investigated in details. In the case of $b \neq 0$, we find that the two-component system (5) possesses the weak kink solution.

The whole paper is organized as follows. In section 2, a Lax pair, bi-Hamiltonian structure as well as infinitely many conservation laws of equation (5) are presented. As a special case of $b = 0$, $v = 2$, a new Lax pair is given for the CH equation. In section 3, the single-peakon, multi-peakon, and two-peakon dynamics are discussed for the case of $b = 0$. Section 4 shows that equation (5) possesses the weak kink solution for the case of $b \neq 0$. Some conclusions and open problems are described in section 5.

2 Lax pair, bi-Hamiltonian structure and conservation laws

Let us consider a pair of linear spectral problems

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}_x = U \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \quad U = \frac{1}{2}\begin{pmatrix} -\alpha & \lambda m \\ -\lambda n & \alpha \end{pmatrix}, \tag{6}$$

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}_t = V \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \quad V = -\frac{1}{2}\begin{pmatrix} A & B \\ C & -A \end{pmatrix}, \tag{7}$$

where $m = u - u_{xx}$, $n = v - v_{xx}$, b is an arbitrary constant, λ is a spectral parameter, $\alpha = \sqrt{1-\lambda^2 b}$, and

$$
\begin{aligned}
A =& \lambda^{-2}\alpha + \frac{\alpha}{2}(uv - u_x v_x) + \frac{1}{2}(uv_x - u_x v),\\
B =& -\lambda^{-1}(u - \alpha u_x) - \frac{1}{2}\lambda m(uv - u_x v_x),\\
C =& \lambda^{-1}(v + \alpha v_x) + \frac{1}{2}\lambda n(uv - u_x v_x).
\end{aligned}
\tag{8}
$$

The compatibility condition of (6) and (7) generates

$$
U_t - V_x + [U, V] = 0. \tag{9}
$$

Substituting the expressions of U and V given by (6) and (7) into (9), we find that (9) is nothing but equation (5). Hence, (6) and (7) exactly give the Lax pair of (5).

Remark 1. As we mentioned above, equation (5) is reduced to the CH equation (1) as $v = 2$. By this reduction, we obtain a new Lax pair of CH equation (1) from (6) and (7):

$$
U = \frac{1}{2}\begin{pmatrix} -\alpha & \lambda m \\ 2\lambda & \alpha \end{pmatrix}, \tag{10}
$$

$$
V = -\frac{1}{2}\begin{pmatrix} \lambda^{-2}\alpha - \alpha u + u_x & -\lambda^{-1}(u - \alpha u_x) + \lambda m u \\ -2\lambda^{-1} + 2\lambda u & -\lambda^{-2}\alpha + \alpha u - u_x \end{pmatrix}. \tag{11}
$$

For $b = 0$ (namely, $\alpha = \sqrt{1-\lambda^2 b} = 1$), that Lax pair reads

$$
\begin{aligned}
U &= \frac{1}{2}\begin{pmatrix} -1 & \lambda m \\ 2\lambda & 1 \end{pmatrix},\\
V &= -\frac{1}{2}\begin{pmatrix} \lambda^{-2} - u + u_x & -\lambda^{-1}(u - u_x) + \lambda m u \\ -2\lambda^{-1} + 2\lambda u & -\lambda^{-2} + u - u_x \end{pmatrix},
\end{aligned}
$$

which is apparently different from the one given by Camassa and Holm in [1]. In other words, here we found a new Lax representation for the Camassa-Holm equation.

Let

$$
\begin{aligned}
K &= \begin{pmatrix} 0 & \partial^2 - 1 \\ 1 - \partial^2 & 0 \end{pmatrix},\\
J &= \begin{pmatrix} \partial m \partial^{-1} m \partial - m \partial^{-1} m & \partial m \partial^{-1} n \partial + m \partial^{-1} n + 2b\partial \\ \partial n \partial^{-1} m \partial + n \partial^{-1} m + 2b\partial & \partial n \partial^{-1} n \partial - n \partial^{-1} n \end{pmatrix}.
\end{aligned}
\tag{12}
$$

By direct calculations, we may verify that J and K are two compatible Hamiltonian operators.

Proposition 1 *Equation (5) can be rewritten as the following bi-Hamiltonian form*

$$(m_t,\ n_t)^T = J\left(\frac{\delta H_1}{\delta m},\ \frac{\delta H_1}{\delta n}\right)^T = K\left(\frac{\delta H_2}{\delta m},\ \frac{\delta H_2}{\delta n}\right)^T, \tag{13}$$

where J and K are given by (12), and

$$\begin{aligned} H_1 &= \frac{1}{2}\int_{-\infty}^{+\infty}(uv + u_x v_x)dx, \\ H_2 &= \frac{1}{4}\int_{-\infty}^{+\infty}[(u^2 v_x + u_x^2 v_x - 2uu_x v)n + 2b(uv_x - u_x v)]dx. \end{aligned} \tag{14}$$

Let us now construct conservation laws of equation (5). Let $\omega = \frac{\phi_2}{\phi_1}$, where ϕ_1 and ϕ_2 are determined through equations (6) and (7). From (6), one can easily verify that ω satisfies the following Riccati equation

$$\omega_x = -\frac{1}{2}\lambda m\omega^2 + \alpha\omega - \frac{1}{2}\lambda n. \tag{15}$$

Equations (6) and (7) give rise to

$$(\ln\phi_1)_x = -\frac{\alpha}{2} + \frac{1}{2}\lambda m\omega, \quad (\ln\phi_1)_t = -\frac{1}{2}\left(A + B\omega\right), \tag{16}$$

which yields conservation law of equation (5):

$$\rho_t = F_x, \tag{17}$$

where

$$\begin{aligned} \rho &= m\omega, \\ F &= \lambda^{-2}(u - \alpha u_x)\omega - \frac{1}{2}\lambda^{-1}\left(\alpha uv - \alpha u_x v_x + uv_x - u_x v\right) + \frac{1}{2}m(uv - u_x v_x)\omega. \end{aligned} \tag{18}$$

Usually ρ and F are called a conserved density and an associated flux, respectively. Let us derive the explicit form of conservation densities for the case of $b = 0$. To this end, we expand ω in terms of negative powers of λ as follows:

$$\omega = \sum_{j=0}^{\infty}\omega_j\lambda^{-j}. \tag{19}$$

Substituting (19) into (15) and equating the coefficients of powers of λ, we arrive at

$$\omega_0 = \sqrt{-\tfrac{n}{m}}, \qquad \omega_1 = \tfrac{mn_x - m_x n - 2mn}{2m^2 n}, \tag{20}$$

and the recursion relation for ω_j:

$$\omega_{j+1} = \tfrac{1}{m\omega_0}\left[\omega_j - \omega_{j,x} - \tfrac{1}{2}m\textstyle\sum_{i+k=j+1,\ i,k\geq 1}\omega_i\omega_k\right], \quad j \geq 1. \tag{21}$$

Inserting (19), (20) and (21) into (18),we finally get the following infinitely many conserved densities and the associated fluxes of equation (5):

$$\begin{aligned} &\rho_0 = \sqrt{-mn}, \quad F_0 = \frac{1}{2}\sqrt{-mn}(uv - u_x v_x), \\ &\rho_1 = \frac{mn_x - m_x n - 2mn}{2mn}, \quad F_1 = -\frac{1}{2}(uv - u_x v_x + uv_x - u_x v) + \frac{1}{2}\rho_1(uv - u_x v_x), \\ &\rho_j = m\omega_j, \quad F_j = (u - u_x)\omega_{j-2} + \frac{1}{2}\rho_j(uv - u_x v_x), \quad j \geq 2, \end{aligned} \tag{22}$$

where ω_j is given by (20) and (21).

3 Peakon solutions for the case of $b = 0$

Let us suppose that the single peakon solution of (5) with $b = 0$ is of the following form

$$u = c_1 e^{-|x-ct|}, \quad v = c_2 e^{-|x-ct|}, \tag{23}$$

where two constants c_1 and c_2 are to be determined. From the expressions of u and v in (23), we see that their first order derivatives are discontinuous at $x = ct$. Thus (23) can not be a solution of equation (5) with $b = 0$ in the classical sense. However, with the help of distribution theory we are able to write out u_x, m and v_x, n as follows

$$\begin{aligned} u_x &= -c_1 sgn(x - ct)e^{-|x-ct|}, \quad m = 2c_1\delta(x - ct), \\ v_x &= -c_2 sgn(x - ct)e^{-|x-ct|}, \quad n = 2c_2\delta(x - ct). \end{aligned} \tag{24}$$

Substituting (23) and (24) into (5) with $b = 0$ and integrating in the distribution sense, one can readily see that c_1 and c_2 should satisfy

$$c_1 c_2 = -3c. \tag{25}$$

In particular, for $c_1 = c_2$, we recover the single peakon solution $u = \pm\sqrt{-3c}e^{-|x-ct|}$ of the cubic CH equation (2) with $b = 0$ [30, 31].

Let us now assume the two-peakon solution as follows:

$$u = p_1(t)e^{-|x-q_1(t)|} + p_2(t)e^{-|x-q_2(t)|}, \quad v = r_1(t)e^{-|x-q_1(t)|} + r_2(t)e^{-|x-q_2(t)|}. \tag{26}$$

In the sense of distribution, we have

$$\begin{aligned} u_x &= -p_1 sgn(x - q_1)e^{-|x-q_1|} - p_2 sgn(x - q_2)e^{-|x-q_2|}, \quad m = 2p_1\delta(x - q_1) + 2p_2\delta(x - q_2), \\ v_x &= -r_1 sgn(x - q_1)e^{-|x-q_1|} - r_2 sgn(x - q_2)e^{-|x-q_2|}, \quad n = 2r_1\delta(x - q_1) + 2r_2\delta(x - q_2). \end{aligned} \tag{27}$$

Substituting (26) and (27) into (5) with $b = 0$ and integrating through test functions yield the following dynamic system:

$$\begin{cases} p_{1,t} = \frac{1}{2}p_1(p_1r_2 - p_2r_1)sgn(q_1 - q_2)e^{-|q_1-q_2|}, \\ p_{2,t} = \frac{1}{2}p_2(p_2r_1 - p_1r_2)sgn(q_2 - q_1)e^{-|q_2-q_1|}, \\ q_{1,t} = -\frac{1}{3}p_1r_1 - \frac{1}{2}\left(p_1r_2 + p_2r_1\right)e^{-|q_1-q_2|}, \\ q_{2,t} = -\frac{1}{3}p_2r_2 - \frac{1}{2}\left(p_1r_2 + p_2r_1\right)e^{-|q_2-q_1|}, \\ r_{1,t} = -\frac{1}{2}r_1(p_1r_2 - p_2r_1)sgn(q_1 - q_2)e^{-|q_1-q_2|}, \\ r_{2,t} = -\frac{1}{2}r_2(p_2r_1 - p_1r_2)sgn(q_2 - q_1)e^{-|q_2-q_1|}. \end{cases} \tag{28}$$

Guided by the above equations, we may conclude the following relations:

$$p_1 = Dp_2, \quad p_1r_1 = A_1, \quad p_2r_2 = A_2, \tag{29}$$

where D, A_1 and A_2 are three arbitrary integration constants.

If $A_1 = A_2$, we arrive at the following solution of (28):

$$\begin{aligned} p_1(t) &= Be^{\frac{1}{2D}(D^2A_1 - A_1)sgn(C_1)e^{-|C_1|}t}, \\ p_2(t) &= \frac{p_1}{D}, \\ r_1(t) &= \frac{A_1}{p_1}, \\ r_2(t) &= \frac{A_1}{p_2}, \\ q_1(t) &= -\left[\frac{1}{3}A_1 + \frac{1}{2D}(D^2A_1 + A_1)e^{-|C_1|}\right]t + \frac{1}{2}C_1, \\ q_2(t) &= q_1(t) - C_1, \end{aligned} \tag{30}$$

where B and C_1 are two arbitrary non-zero constants. In this case, the collision between two peakons will not happen since $q_2(t) = q_1(t) - C_1$.

In particular, as $A_1 = B = D = 1$, $C_1 = 2$, (30) is reduced to

$$\begin{aligned} &p_1(t) = p_2(t) = r_1(t) = r_2(t) = 1, \\ &q_1(t) = -\left(\frac{1}{3} + e^{-2}\right)t + 1, \qquad q_2(t) = -\left(\frac{1}{3} + e^{-2}\right)t - 1. \end{aligned}$$

Thus, the associated solution of (5) with $b = 0$ becomes

$$u(x,t) = v(x,t) = e^{-|x+(\frac{1}{3}+e^{-2})t-1|} + e^{-|x+(\frac{1}{3}+e^{-2})t+1|}, \tag{31}$$

which has two peaks, and looks like a M-shape soliton solution [26], but not, and see Figure 1 for this M-shape two-peakon solution.

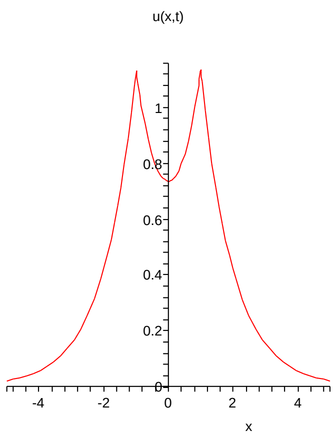


Figure 1: The M-shape two-peakon solution $u(x,t)$ in (31) at the moment of $t=0$.

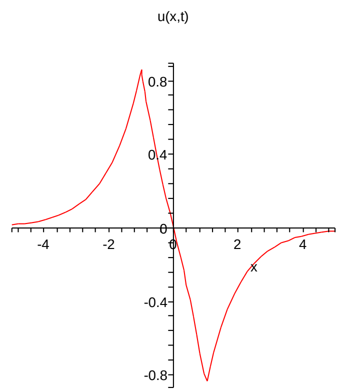


Figure 2: The N-shape peak-trough solution $u(x,t)$ in (32) at the moment of $t=0$.

As $A_1=-B=-D=1$, $C_1=2$, the associated solution of (5) with $b=0$ becomes

$$u(x,t)=v(x,t)=-e^{-\left|x+\left(\frac{1}{3}-e^{-2}\right)t-1\right|}+e^{-\left|x+\left(\frac{1}{3}-e^{-2}\right)t+1\right|}, \tag{32}$$

which has one peak and one trough and looks like N-shape soliton solution. See Figure 2 for this N-shape two-peakon (peakon-antipeakon interaction) solution.

As $B=2D=1$, $A_1=C_1=2$, (30) becomes

$$\begin{aligned} p_1(t)&=\frac{1}{2}p_2(t)=e^{-\frac{3}{2}e^{-2}t}, \quad r_1(t)=2r_2(t)=2e^{\frac{3}{2}e^{-2}t},\\ q_1(t)&=-\left(\frac{2}{3}+\frac{5}{2}e^{-2}\right)t+1, \qquad q_2(t)=-\left(\frac{2}{3}+\frac{5}{2}e^{-2}\right)t-1. \end{aligned} \tag{33}$$

and the associated solution of (5) with $b=0$ becomes

$$\left\{\begin{array}{l} u(x,t)=e^{-\frac{3}{2}e^{-2}t}\left(e^{-\left|x+\left(\frac{2}{3}+\frac{5}{2}e^{-2}\right)t-1\right|}+2e^{-\left|x+\left(\frac{2}{3}+\frac{5}{2}e^{-2}\right)t+1\right|}\right),\\ v(x,t)=e^{\frac{3}{2}e^{-2}t}\left(2e^{-\left|x+\left(\frac{2}{3}+\frac{5}{2}e^{-2}\right)t-1\right|}+e^{-\left|x+\left(\frac{2}{3}+\frac{5}{2}e^{-2}\right)t+1\right|}\right). \end{array}\right. \tag{34}$$

From (33), one can easily see that the amplitudes $p_1(t)$ and $p_2(t)$ of potential $u(x,t)$ are two monotonically decreasing functions of t, while the amplitudes $r_1(t)$ and $r_2(t)$ of potential $v(x,t)$ are two monotonically increasing functions of t. Figures 3 and 4 show the profiles of the potentials $u(x,t)$ and $v(x,t)$.

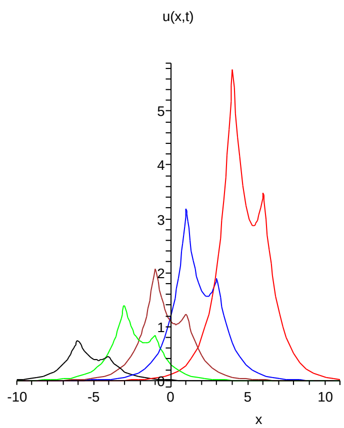


Figure 3: The two-peakon solution $u(x,t)$ in (34). Red line: $t=-5$; Blue line: $t=-2$; Brown line: $t=0$; Green line: $t=2$; Black line: $t=5$.

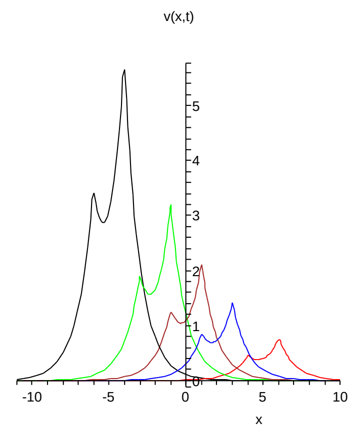


Figure 4: The two-peakon solution $v(x,t)$ in (34). Red line: $t=-5$; Blue line: $t=-2$; Brown line: $t=0$; Green line: $t=2$; Black line: $t=5$.

If $A_1 \neq A_2$, we may obtain the following solution of (28):

$$
\begin{aligned}
p_1(t) &= Be^{\frac{3(A_2D^2-A_1)}{2D(A_1-A_2)}e^{-\frac{1}{3}|(A_1-A_2)t|}},\\
p_2(t) &= \frac{p_1}{D},\\
r_1(t) &= \frac{A_1}{p_1},\\
r_2(t) &= \frac{A_2}{p_2},\\
q_1(t) &= -\frac{1}{3}A_1 t + \frac{3(A_2D^2+A_1)}{2D(A_1-A_2)} sgn[(A_1-A_2)t]\left(e^{-\frac{1}{3}|(A_1-A_2)t|}-1\right),\\
q_2(t) &= -\frac{1}{3}A_2 t + \frac{3(A_2D^2+A_1)}{2D(A_1-A_2)} sgn[(A_1-A_2)t]\left(e^{-\frac{1}{3}|(A_1-A_2)t|}-1\right),
\end{aligned}
\tag{35}
$$

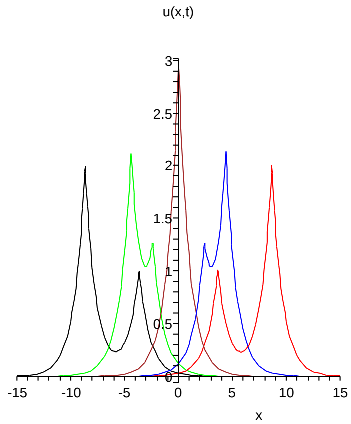
where B is an arbitrary integration constant. Let us study the following special cases of this solution.

Case 1. Let $A_1=1$, $A_2=4$, $B=1$, $D=\frac{1}{2}$, then

$$
\left\{
\begin{array}{l}
p_1(t)=r_1(t)=1, \quad p_2=r_2(t)=2,\\
q_1(t)=-\frac{1}{3}t+2sgn(t)\left(e^{-|t|}-1\right),\\
q_2(t)=-\frac{4}{3}t+2sgn(t)\left(e^{-|t|}-1\right).
\end{array}
\right.
\tag{36}
$$

The associated two-peakon solution of (5) becomes

$$
u(x,t)=v(x,t)=e^{-\left|x+\frac{1}{3}t-2sgn(t)\left(e^{-|t|}-1\right)\right|}+2e^{-\left|x+\frac{4}{3}t-2sgn(t)\left(e^{-|t|}-1\right)\right|}. \tag{37}
$$



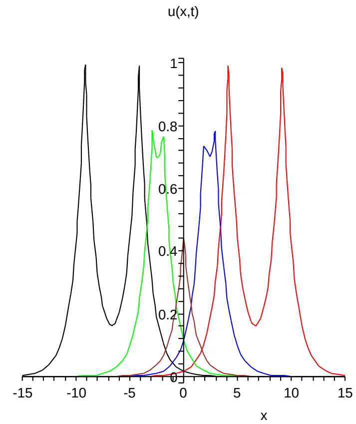


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Figure 5: The two-peakon solution $u(x,t)$ in (37). Red line: $t=-5$; Blue line: $t=-2$; Brown line: $t=0$ (collision); Green line: $t=2$; Black line: $t=5$.

Figure 6: The two-peakon solution $u(x,t)$ in (39). Red line: $t=-5$; Blue line: $t=-1$; Brown line: $t=0$ (collision); Green line: $t=1$; Black line: $t=5$.

As $t<0$ and t is going to 0, the tall peakon with amplitude 2 and at peak position q_2 chases after the short peakon with the amplitude 1 and at peak position q_1. The two-peakon collides at the moment of $t=0$. After the collision ($t>0$), the peaks separate (the tall peakon surpasses the short one) and develop on their own way. See Figure 5 for the detailed development of this kind of two-peakon.

Case 2. Let $A_1=1$, $A_2=4$, $B=1$, $D=1$, then we have

$$\left\{\begin{array}{l} p_1(t)=p_2(t)=e^{-\frac{3}{2}e^{-|t|}}, \\ r_1(t)=e^{\frac{3}{2}e^{-|t|}}, \quad r_2(t)=4e^{\frac{3}{2}e^{-|t|}}, \\ q_1(t)=-\frac{1}{3}t+\frac{5}{2}sgn(t)\left(e^{-|t|}-1\right), \\ q_2(t)=-\frac{4}{3}t+\frac{5}{2}sgn(t)\left(e^{-|t|}-1\right). \end{array}\right. \tag{38}$$

The associated two-peakon solution of (5) becomes

$$\left\{\begin{array}{l} u(x,t)=e^{-\frac{3}{2}e^{-|t|}}\left(e^{-\left|x+\frac{1}{3}t-\frac{5}{2}sgn(t)\left(e^{-|t|}-1\right)\right|}+e^{-\left|x+\frac{4}{3}t-\frac{5}{2}sgn(t)\left(e^{-|t|}-1\right)\right|}\right), \\ v(x,t)=e^{\frac{3}{2}e^{-|t|}}\left(e^{-\left|x+\frac{1}{3}t-\frac{5}{2}sgn(t)\left(e^{-|t|}-1\right)\right|}+4e^{-\left|x+\frac{4}{3}t-\frac{5}{2}sgn(t)\left(e^{-|t|}-1\right)\right|}\right). \end{array}\right. \tag{39}$$

For the potential $u(x,t)$, the two-peakon solution possesses the same amplitude $e^{-\frac{3}{2}e^{-|t|}}$, which reaches the minimum value at the moment of collision ($t=0$). Figure 6 shows the profile of the two-peakon dynamics for the potential $u(x,t)$. For the potential $v(x,t)$, the two-peakon solution with the amplitudes $e^{\frac{3}{2}e^{-|t|}}$ and $4e^{\frac{3}{2}e^{-|t|}}$ collides at the moment of $t=0$. At this moment, the amplitudes attain the maximum value and the two-peakon overlaps into one peakon $5e^{\frac{3}{2}}e^{-|x|}$, which is much higher than other moments. See Figures 7 and 8 for the 2-dimensional and 3-dimensional graphs of the two-peakon dynamics for the potential $v(x,t)$.

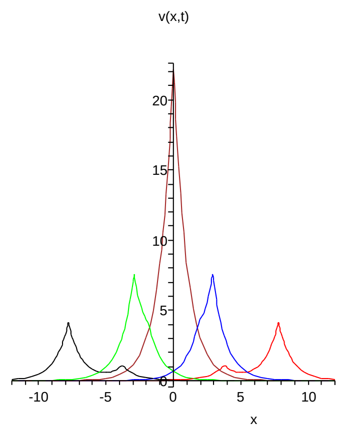

Figure 7: The two-peakon solution $v(x,t)$ in (39). Red line: $t=-4$; Blue line: $t=-1$; Brown line: $t=0$ (collision); Green line: $t=1$; Black line: $t=4$.

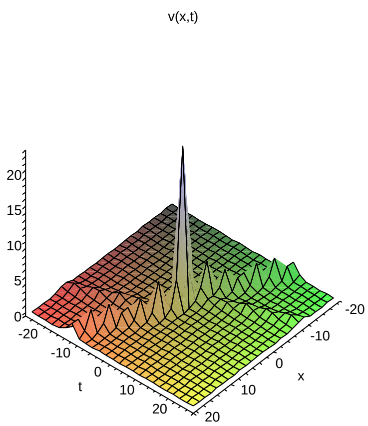

Figure 8: 3-dimensional graph for the two-peakon solution $v(x,t)$ in (39).

Case 3. Let $A_1=1$, $A_2=4$, $B=1$, $D=-1$, then we have

$$\left\{\begin{array}{l} p_1(t)=-p_2(t)=e^{\frac{3}{2}e^{-|t|}}, \\ r_1(t)=e^{-\frac{3}{2}e^{-|t|}}, \quad r_2(t)=-4e^{-\frac{3}{2}e^{-|t|}}, \\ q_1(t)=-\frac{1}{3}t-\frac{5}{2}sgn(t)\left(e^{-|t|}-1\right), \\ q_2(t)=-\frac{4}{3}t-\frac{5}{2}sgn(t)\left(e^{-|t|}-1\right). \end{array}\right. \tag{40}$$

The associated two-peakon solution of (5) becomes

$$\left\{\begin{array}{l} u(x,t)=e^{\frac{3}{2}e^{-|t|}}\left(e^{-\left|x+\frac{1}{3}t+\frac{5}{2}sgn(t)\left(e^{-|t|}-1\right)\right|}-e^{-\left|x+\frac{4}{3}t+\frac{5}{2}sgn(t)\left(e^{-|t|}-1\right)\right|}\right), \\ v(x,t)=e^{-\frac{3}{2}e^{-|t|}}\left(e^{-\left|x+\frac{1}{3}t+\frac{5}{2}sgn(t)\left(e^{-|t|}-1\right)\right|}-4e^{-\left|x+\frac{4}{3}t+\frac{5}{2}sgn(t)\left(e^{-|t|}-1\right)\right|}\right). \end{array}\right. \tag{41}$$

For the potential $u(x,t)$, the peakon-antipeakon collides and vanishes at the moment of $t=0$. After the collision, the peakon and antipeakon reemerge and separate. For the potential $v(x,t)$, the peakon and trough overlap at the moment of $t=0$, and then they separate. Figures 9 and 10 show the peakon-antipeakon dynamics for the potentials $u(x,t)$ and $v(x,t)$.

In general, we suppose N-peakon solution has the following form:

$$u(x,t)=\sum_{j=1}^{N}p_j(t)e^{-|x-q_j(t)|}, \quad v(x,t)=\sum_{j=1}^{N}r_j(t)e^{-|x-q_j(t)|}. \tag{42}$$

Substituting (42) into (5) with $b=0$ and integrating through test functions, we obtain the N-peakon dynamic system as follows:

$$\left\{\begin{array}{l} p_{j,t}=\frac{1}{2}p_j\sum_{i,k=1}^{N}p_ir_k\left(sgn(q_j-q_k)-sgn(q_j-q_i)\right)e^{-|q_j-q_k|-|q_j-q_i|}, \\ q_{j,t}=\frac{1}{6}p_jr_j-\frac{1}{2}\sum_{i,k=1}^{N}p_ir_k\left(1-sgn(q_j-q_i)sgn(q_j-q_k)\right)e^{-|q_j-q_i|-|q_j-q_k|}, \\ r_{j,t}=-\frac{1}{2}r_j\sum_{i,k=1}^{N}p_ir_k\left(sgn(q_j-q_k)-sgn(q_j-q_i)\right)e^{-|q_j-q_k|-|q_j-q_i|}. \end{array}\right. \tag{43}$$

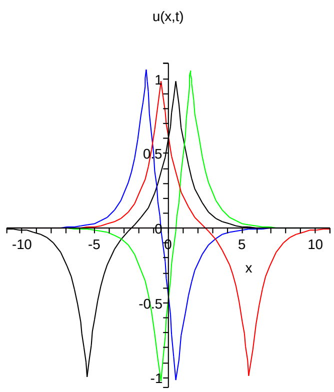

Figure 9: Peakon-antipeakon solution $u(x,t)$ in (41). Red line: $t=-6$; Blue line: $t=-2$; At $t=0$ they collide and vanish; Green line: $t=2$; Black line: $t=6$.

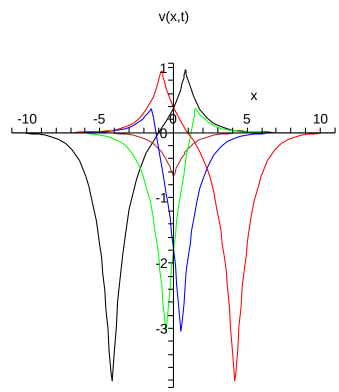

Figure 10: Peakon-antipeakon solution $v(x,t)$ in (41). Red line: $t=-5$; Blue line: $t=-2$; Brown line: $t=0$ (collision); Green line: $t=2$; Black line: $t=5$.

It is interesting to study whether the above system is able to be rewritten as an integrable Hamiltonian system by introducing a Poisson bracket. We will investigate this in the future.

4 Weak kink solution for the case of $b \neq 0$

In this section, we show that equation (5) with $b \neq 0$ possesses a weak kink solution. We assume equation (5) has the following kink wave solution:

$$u = C_1 sgn(x-ct)\left(e^{-|x-ct|}-1\right), \quad v = C_2 sgn(x-ct)\left(e^{-|x-ct|}-1\right), \tag{44}$$

where the constant c is the wave speed, and C_1 and C_2 are two constants to be determined. In fact, if $C_1 \neq 0$ and $C_2 \neq 0$, the potentials u and v in (44) are kink wave solutions due to

$$\begin{aligned} &\lim_{x\to+\infty} u = -\lim_{x\to-\infty} u = -C_1, \\ &\lim_{x\to+\infty} v = -\lim_{x\to-\infty} v = -C_2. \end{aligned} \tag{45}$$

One may easily check that the first order partial derivatives of (44) read

$$\begin{aligned} u_x &= -C_1 e^{-|x-ct|}, \quad u_t = cC_1 e^{-|x-ct|}, \\ v_x &= -C_2 e^{-|x-ct|}, \quad v_t = cC_2 e^{-|x-ct|}. \end{aligned} \tag{46}$$

But, unfortunately, The second and higher order partial derivatives of (44) do not exist at the peak point $x=ct$. Thus, like the case of peakon solutions, the kink wave solution in the form of (44) should also be understood in the distribution sense, and therefor we call (44) the weak kink solution to equation (5) with $b \neq 0$.

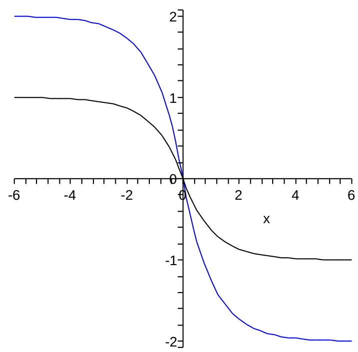

Figure 11: The weak kink solution at $t = 0$. Black line: $u(x, 0)$; Blue line: $v(x, 0)$.

Substituting (44) and (46) into (5) and integrating through test functions, we may arrive at

$$\begin{cases} c = -\frac{1}{2}b, \\ C_1C_2 = -b. \end{cases} \tag{47}$$

In the above formula, $c = -\frac{1}{2}b$ means that the kink wave speed is exactly $-\frac{1}{2}b$. In particular, we take $b = -2$ and $C_1 = 1$, then the corresponding weak kink solution is cast to

$$u = sgn(x-t)\left(e^{-|x-t|} - 1\right), \quad v = 2sgn(x-t)\left(e^{-|x-t|} - 1\right). \tag{48}$$

See Figure 11 for the profile of the kink wave solution.

Usually, multi-peakon solutions take the form of superpositions of the single-peakon solutions. But, we need to point out that equation (5) with $b \neq 0$ does not allow the multi-kink solution in the form of the superpositions of single-kink solutions like this:

$$u = \sum_{j=1}^{N} p_j(t)sgn(x - q_j(t))\left(e^{-|x-q_j(t)|} - 1\right), \quad v = \sum_{j=1}^{N} r_j(t)sgn(x - q_j(t))\left(e^{-|x-q_j(t)|} - 1\right). \tag{49}$$

In fact, substituting (49) into (5) and integrating through test functions, we find that the solution assumed in the form (49) is reduced to zero or single-kink solution (44). In paper [31], we proposed the peakon-kink interactional solutions in the form of

$$u = p_0(t)sgn(x - q_0(t))\left(e^{-|x-q_0(t)|} - 1\right) + \sum_{j=1}^{N} p_j(t)e^{-|x-q_j(t)|},$$

for the cubic CH equation (2) (see section 4.2 in [31]). Here, we naturally hope to seek the peakon-kink interactional solutions for the two-component cubic equation (5) with $b \neq 0$ in the

form of

$$\begin{cases} u = p_0(t)sgn(x - q_0(t))\left(e^{-|x-q_0(t)|} - 1\right) + \sum_{j=1}^{N} p_j(t)e^{-|x-q_j(t)|}, \\ v = r_0(t)sgn(x - q_0(t))\left(e^{-|x-q_0(t)|} - 1\right) + \sum_{j=1}^{N} r_j(t)e^{-|x-q_j(t)|}. \end{cases} \tag{50}$$

Unfortunately, by direct calculations, we find that such peakon-kink interactional solutions in the form (50) exist only when the potentials u and v satisfy the reduction condition $u = kv$, which reduces the two-component cubic system (5) to the cubic CH equation (2). In other words, we have not yet found multi-kink wave solutions and peakon-kink interactional solutions to the two-component cubic CH system (5) for the general case: $u \neq kv$. This is an interesting and challenging topic, and we will make a further study elsewhere.

5 Conclusions and discussions

In our paper, we present a new two-component integrable system (5) with cubic nonlinearity and provide its Lax representation, bi-Hamiltonian structure, and infinitely many conservation laws. The peakon solutions for the system are derived for the case of $b = 0$. In particular, the two-peakon dynamical system is explicitly solved. The M-shape and N-shape peakons are analytically obtained, and the collisions between two peakons are discussed and shown through their graphs. We also find that this system admits weak kink solutions for the case of $b \neq 0$. Other topics, such as smooth soliton solutions, cuspons, peakon stability, and algebra-geometric solutions, remain to be developed for our new two-component integrable system (5).

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